RESTRICTED

RESTRICTED

NEW YORK UNIVERSITY
WASHINGTON SQUARE COLLEGE
MATHEMATICS DEPARTMENT RESEARCH GROUP

FINAL REPORT

SEPTEMBER, 1948

Contract No. W 28-099-ac-172
with
Watson Laboratories, Air Materiel Command
United States Air Force

RESTRICTED

RESTRICTED

New York University

Washington Square College

Mathematics Department Research Group

FINAL REPORT

September, 1948

No. 19

Contract No. W 28-099-ac-172

with

Watson Laboratories

Air Materiel Command

United States Air Force

Subject:

The propagation characteristics of very short waves, including problems of reflection, refraction and interaction.

Morris Kline

Morris Kline
Project Director

CONTENTS

Article	Page
I. Introduction	1
II. Propagation in Stratified Media.	1
III. The Scattering of Radiation by an Obstacle.	5
IV. Asymptotic Developments of Steady State Electromagnetic Fields.	11
V. Inversion Method and Perturbation Method	21
VI. Wiener-Hopf Methods in Propagation Problems.	22
VII. Bibliography	22
VIII. Education	23
IX. Administrative Report (for 1 July to 30 September)	
1. Personnel	23
2. Correspondence	23
3. Conferences	24
X. Research Reports Nos. 172-1 to 172-9	25

I. Introduction

This is a summary of work performed under Contract No. W28-099-ac-172 awarded by the Watson Laboratories of the United States Air Force to a group of mathematicians associated with the Institute for Mathematics and Mechanics of New York University. The group began to be organized in July, 1946 to conduct basic mathematical research in the field of transmission, reflection and diffraction of high frequency electromagnetic waves. The investigations to be carried out were conceived as primarily theoretical in nature. However, it was a guiding thought that the studies be advanced, where possible, to a point of potential engineering applicability.

A number of Research Reports were submitted by the group to the Watson Laboratories. They are discussed in this report, and are listed by title and author at the conclusion. Reference to them, in the report, is made by the occurrence of numerals in square brackets.

The plan of this report is as follows. The next section will deal with work on propagation in stratified media, Part III with the scattering of propagation by obstacles. Part IV will give a rather detailed account of work in progress, on a general theory of the relation of geometric optics to electromagnetism. Insofar as this work is in part directed to the discovery of asymptotic expressions for certain integrals it represents a continuation of the work described in II where such integrals are used (see item 3 of II). However, the approximate evaluation of these complex integrals is only a connecting link between two distinct problems. In V there are brief indications of other work of the group, some of which came to be abandoned at least for the time being.

II. Propagation in Stratified Media.

One of the principal problems studied by members of the group was the propagation of electromagnetic radiation in a space with variable ε, μ, and σ. This problem becomes approachable only if it is restricted by requiring that the electromagnetic parameters be functions of a single space coordinate. In the case of a plane-stratified medium, for example, one assumes that ε, μ, and σ vary with one coordinate only, and the medium is characterized by the assignment of three functions, $\varepsilon(z)$, $\mu(z)$, $\sigma(z)$, where z is one of the rectangular space-

coordinates. A problem is now defined by the specification of a source of electromagnetic radiation, e.g., a harmonic dipole located at the origin of coordinates. One then seeks formulae which describe the radiation field everywhere in space. This class of problems embraces the so-called problem of "anomalous propagation": i.e., the transmission of radar intelligence through the atmosphere (account being taken, in some manner, of the sphericity of the earth).

This problem has received a great deal of attention, particularly in recent years when its importance in radar communication became evident, and many references to it will be found in the Bibliography [9] prepared by members of this group. In spite of the brilliant work already devoted to this problem, the great importance of the problem from a theoretical as well as a practical point of view calls for further effort, particularly in view of the extremely formidable numerical calculations to which this work gives rise. Moreover, the present solutions utilize approximations which seriously limit the range of applicability of the results.

1. In the first Report [3] submitted on this general subject, it is assumed that the half-spaces $z < 0$ and $z > \ell$ (a positive constant) are occupied by homogeneous substances, and that in the remaining space, $0 < z < \ell$, there are given three functions $\varepsilon(z)$, $\mu(z)$, $\sigma(z)$ which are piecewise differentiable in the interval 0 to ℓ. It is assumed that a plane wave approaches the inhomogeneous slab from below, and the problem is to investigate the effect which the layer system has on the propagation of the wave.

This problem is approached through the special case, itself of great interest, of a <u>finite</u> <u>multilayer</u>. Here, the interval 0 to ℓ is supposed divided into a finite number of intervals: $0 = \ell_0 < \ell_1 < \ell_2 < \dots < \ell_n = \ell$ in each of which $\varepsilon(z)$, $\mu(z)$, $\sigma(z)$ are constant. For this case there is derived in the paper a complete set of recursion formulas ([3] , p.17, formulas 25 and 26) permitting the ready calculation of all desired field quantities. The case that ε, μ, σ are piecewise continuously differentiable, is then treated by a passage to a limit. Here the reflection and transmission coefficients of the layer system are expressed relatively simply in terms of certain functions, which, however are determined by

Riccati equations whose variable coefficients depend on the parameter functions $\varepsilon(z)$, $\mu(z)$ and $\sigma(z)$.

2. A second Report [4] considerably generalized the multilayer case mentioned above by introducing a dipole source of radiation. The first part of the paper investigates the case of a multilayer contained between two semi-infinite homogeneous media with a stream of plane waves of constant amplitude impinging on the layer from below, the waves being so polarized that the magnetic vector is parallel to the multilayer. Next a dipole source below the multilayer is treated, the source being represented as a superposition of plane waves. Finally, the dipole is located within the multilayer. The special case of a single layer is worked out in detail. The radiation fields are expressed in the form of contour integrals, which take account of the representation of the dipole source as a superposition of plane waves.

The methods of the paper raise certain difficult questions of a mathematical nature concerning the convergence of the integral representations. This question, as well as the problem of effectively calculating with the form of solution given in the paper were postponed in favor of the more general investigation described below.

3. In a fundamental paper [6] the propagation of electromagnetic waves through a quite general plane stratified inhomogeneous medium was studied for the case of an arbitrary source. It was shown that the problem may be reduced to the study of a pair of scalar functions, each subject to a partial differential equation, the two equations differing only in the interchange of $\varepsilon(z)$ and $\mu(z)$, where these occur ([6] p.12, formula 4.7). For fields periodic in time, the problem is then reduced to the superposition of solutions of differential equations of Riccati type ([6] p. 32 formulas (7.10) and (7.11)). This formulation has the advantage that the solution functions of the Riccati equations express very directly the distortion in wave front and the change in amplitude of a plane wave incident on a stratified medium. The total electromagnetic field in the medium is expressed as a complex integral involving the representation of the arbitrary source as a superposition of plane waves, and the solution functions of the Riccati equation ([6], formulas (7.8) and (7.9)).

The formulas of this paper constitute a solution of the general problem in the sense that it is possible to apply to them numerical procedures to calculate the fields associated with a specified source and chosen set of parameter functions $\varepsilon(z)$, $\mu(z)$, $\sigma(z)$. However, some initial efforts in this direction indicate that the task would be an extremely formidable one, and there is no reason to believe, as yet, that such an undertaking would offer advantages over the current procedure of "normal-mode" calculations based on methods of Furry, Pekeris, and others. (See, among many other references listed in the Bibliography [9], The Radiation Laboratory Report #680, Theory of Characteristic Functions in Problems of Anomalous Propagation, February 28, 1945).

It is felt that much further work of a theoretical nature must first be carried through in connection with approximate evaluations of the complex integrals appearing in this paper, and it is hoped that the project described in Part IV of this report may supply some of the methods for such an evaluation.

4. Since the initial motivation for the studies described above was the problem of "anomalous propagation", some thought has been given to the problem of adapting a theory for a plane-stratified medium to the case of the earth. This is conventionally handled by the use of a modified index of refraction. The problem is discussed in the Introduction to [6], reported on above, and in [8]. It will be shown in a later Report that the same modified index of refraction can be applied to the "flat-earth" theory of [6]. However, this device is not satisfactory in some applications, and it is desirable that the problem be approached in some other way. The most direct approach would seem to be an investigation of spherically stratified media.

One such study is begun in [8] which deals with a spherically symmetric medium. Here the electromagnetic parameters depend only on the distance to the origin of space. This study parallels the first part of a preceding Report [6] by showing that the components of an electromagnetic field in a medium whose structural constants depend only on the distance to the origin can be derived from two scalar functions. The use of a special curvilinear coordinate system simplifies the form of the differential equations which govern these scalar functions ([8] p.16, (5.9)), and makes easier the calculation of the components of the electromagnetic field (ibid.(5.10)). However, at the point at which this Report stops, the differential

equations are not explicitly solved. The Report brings out the striking similarities between the equations associated with the spherically stratified and the plane stratified media. It is hoped that some advantage can be taken of this.

5. Another approach to the problem of anomalous propagation over a spherical earth is now well under way carrying over to the case of spherical stratification the method of characteristic functions (see the previously cited Radiation Laboratory Report) now employed for a flat earth and horizontally stratified atmosphere. The object of this investigation will be to determine whether this more direct approach to the problem of "anomalous propagation" in the earth's atmosphere will substantially complicate the task of numerical calculation associated with the "flat-earth" theory.

6. Reference has already been made to work in progress on the problem of the numerical calculation of anomalous propagation fields on the basis of the formalism developed in [6]. The work is being undertaken by one of the members of the group well acquainted with calculation procedures used in conjunction with modern computing instrumentation. The object of the study is to determine whether direct computation of the integrals found in [6] seems practically feasible. Results so far are inconclusive, and the investigation is being continued.

III. The Scattering of Radiation by an Obstacle.

The general problem of scattering of radiation, i.e., the reflection from, transmission through and diffraction by an arbitrary obstacle, can be subsumed under the larger topic of propagation in non-homogeneous media. However, the problem is here construed in the narrower sense that an obstacle of one kind of material is imbedded in an otherwise homogeneous, electrically isotropic space in which there is present some source of electromagnetic radiation, far removed from the source. The condition of large distance of the source from the obstacle is made to insure that the field reflected back to the source is negligible in comparison with the field initiated there. The problem, given the nature of the source, the medium, and the obstacle, and the geometry of the configuration: source and obstacle, is to find the total field at all points of space. This problem can be solved exactly only in cases where the obstacle is of particularly simple

geometric character. In recent years several papers have appeared which have investigated the problem for the case of fairly arbitrary obstacles, using approximate methods (see references in [5], p. 48). Approximate methods of a very general application to problems of this kind consist in the methods of geometrical optics, corresponding to the limit of zero wave length, and the Kirchhoff method, which will be briefly indicated below. Since these methods rest upon approximations to electromagnetic theory, results obtained by them ought to be checked against methods which are more accurate in the boundary conditions imposed upon the fields at the surface of the obstacle. Such an investigation is undertaken in [5], discussed below.

1. However, in the first Report [1] on this general problem the method used is the Kirchhoff method, and some of the results obtained by it are checked against the method of geometrical optics. The obstacle is assumed to be perfectly reflecting, but of arbitrary smooth shape. The wave equation satisfied by the total vector field is transformed to an integro-differential equation. In general, this equation involves both surface and volume integrals. In the present case of a perfectly reflecting obstacle, the volume integral is absent. Now the essential feature of the Kirchhoff method is invoked. This is the assumption of boundary conditions which are false in the electromagnetic case, but plausible in geometrical optics: e.g., it is assumed that the total field is double the incident field on the geometrically illuminated part of the surface, and zero elsewhere on the surface. The effect of these assumptions is to reduce the problem of finding the total field to the problem of evaluating a definite surface integral. This evaluation, while theoretically possible given the geometric shape of the obstacle, is prohibitively difficult to perform exactly. An approximate evaluation is found, essentially by the method of stationary phase applied to surfaces.

Effectively, the succeeding Report [5] is of greater generality, and a discussion of the nature of the results obtained here [1] will be postponed to the next section.

2. It is possible to set up a pair of integro-differential equations for the electric and magnetic fields in the presence of a general obstacle, without the restriction that the obstacle be perfectly reflecting. But without some restriction on the obstacle, these equations are completely unmanageable. It was observed in the acoustic case by H. Primakoff and J. B. Keller (see ref. 12 in [5]) and carried over to the electromagnetic case in [5] that these equations can be reduced for smooth curved obstacles of constant thickness h, if h is sufficiently small. In this case, the volume integrals may be replaced by surface integrals, and certain differentiations of field quantities can be carried out explicitly.

The boundary conditions imposed are those of electromagnetism, so that the approximations made in geometrical optics or in the Kirchhoff method are now avoided. There remains the approximation, as in [1] that a certain surface integral must be evaluated by some approximate method. The integral is of the general form:

$$\int_S f(r') \, e^{i(k_1|r'-r_1| + k_2|r'-r_2|)} \, ds$$

where r' is the vector position of a variable point on the surface S, r_1 and r_2 locate the source point and observation point, $f(r')$ is a certain function slowly varying over the surface in comparison with the exponential, and k_1 and k_2 are certain constants. This can be evaluated by an adaptation of the method of stationary phase.

The solution fields are obtained in the form of an expansion in powers of h and the first two terms can be explicitly calculated by the methods of this paper. The first term is merely the incident field, i.e. the field due to the source in the absence of any obstacle. The first correction term to the incident field is shown to equal the incident field of the source multiplied by three factors, designated as the geometric, the phase, and the reflectivity factors. The first of these, in the limit of zero wave length agrees with the reflection coefficient obtained by geometrical optics, and the second with a corresponding phase factor which results from an application of the Kirchhoff method. The third factor is not present in the Kirchhoff solution and represents an actual improvement in the accuracy of the solution.

The paper [5] discusses the physical significance of these factors in considerable detail and compares them with results previously obtained by other methods, (see, in particular §5 , ibid., pp. 26-35). The agreement, in all cases, appears to be satisfactory. As a further check upon the theory the explicit formulas were derived for the exact fields corresponding to the geometrically simple case of a spherical shell with a radiating dipole at the center. This is discussed below.

3. In a subsequent Report [7] explicit expressions are obtained for the exact fields everywhere in space associated with a radiating dipole at the center of a sphere of one medium encased in a spherical shell of medium 2 which is bounded by medium 3. The electromagnetic parameters are constant within each medium, but otherwise arbitrary, and the shell radii are arbitrary. The reflected and transmitted fields are examined in various special cases and a variety of results known for flat media are obtained by limiting considerations applied to the shell radii.

Comparison is made of the exact results of this special case with the approximations obtained by the general theory above [5]. It is found that except for one field component and except near the singularity of the reflected field, the approximate solution is good.

4. A study is under way of the general problem of effectively calculating by the formulas of [5] the fields associated with a particular source and obstacle. Since the formulas in [5] are quite complicated, necessarily so as one sees by considering the complexity of the exact fields associated with the simple special case investigated in [7] , it is natural to inquire how useful these formulas may be for quantitative results. The study is confined to the case of quadric surfaces and special positions of the radiating source.

5. Another study in progress seeks to develop formulas for the case of an interface of arbitrary shape between two semi-infinite media. This work was in abeyance in the past months; it is described in some detail in the preceding Quarterly Progress Report.

6. A quite different approach to scattering problems is nearing completion, using variational methods. This project is devoted to calculating the back-scattering cross-section, that is, the power radiated back when a plane wave of unit power density per unit area impinges on a target, for a variety of obstacles.

The differential equations and exact boundary conditions of the problem are converted to integral equations. It is shown how the problem of solving these equations may be transformed into the problem of finding extreme values of certain integrals. This problem is treated by the methods of the calculus of variations.

The first problem treated in this way is the scattering of sound by rigid bodies of arbitrary shape. Here the back-scattering is found directly as the extreme value of the ratio of two integrals. Examples are calculated for certain simple shapes where exact answers can be obtained in order to estimate the accuracy of the method of the paper. Another problem treated is the scattering of electric waves by cylindrical obstacles where the cross section of the cylinder may be arbitrary. Finally, it is indicated what other problems may be treated by the methods of the paper.

To bring out the details of the method one may consider the scattering of sound by obstacles of arbitrary shape. Denote the velocity potential by $\varphi(x,y,z)\, e^{-ikct}$, which satisfies the wave equation:

$$1) \quad (\nabla^2 + k^2)\, \varphi(x,y,z) = 0$$

The obstacle is taken as rigid and so the normal derivative of φ vanishes on it. Assume that the field incident on the obstacle may be represented as an incoming plane wave, propagated in the positive x-direction. The total field contains, also, an outgoing spherical wave, whose angular pattern is to be determined. Let the value of φ on the obstacle be denoted by $\varphi(s)$. Then it is shown that φ satisfies the integral equation:

$$2) \quad \varphi(x,y,z) = e^{ikx} - \frac{1}{4\pi} \int_{S'} \varphi(s') \frac{\partial}{\partial n'} \frac{e^{ik|r-r'|}}{|r-r'|}\, ds'$$

Here the integral extends over the surface of the obstacle, and $|r-r'|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$. The symbol $\frac{\partial}{\partial n'}$ denotes the normal derivative. The condition that the normal derivative of φ vanishes on the obstacle

gives the integral equation, valid on the obstacle:

$$3) \qquad 0 = \frac{\partial}{\partial n} e^{ikx}\Big|_S - \frac{1}{4\pi}\int_{S'} \varphi(s') \frac{\partial}{\partial n}\Big|_S \frac{\partial}{\partial n'} \frac{e^{ik|r-r'|}}{|r-r'|} ds'$$

Here S and S' refer to the surface of the obstacle, the primed co-ordinates refer to a running point on the obstacle, the unprimed to a fixed point.

This is an integral equation of the form:

$$4) \qquad H(s) = \int_{S'} \varphi(s') K(s,s') ds'$$

The general theorem can be proved that the function which renders stationary the integral expression:

$$5) \quad I(\varphi) = \frac{\int_S \int_{S'} \varphi(s) K(s,s') \varphi(s') ds \, ds'}{\left[\int_S \varphi(s) H(s) ds\right]^2}$$

is the function which satisfies the integral equation 4) save for a multiplicative constant which may be used to normalize φ to unit incident power density. Consequently, the equation 3) may be expressed in the variational form 5).

If we compute the asymptotic form of φ from the integral expression 2) we obtain:

$$6) \quad \varphi \rightarrow e^{ikx} + f(\theta,\varphi) \frac{e^{ikr}}{r} \quad .$$

Here r, θ, φ are spherical coordinates corresponding to x,y,z. The function $f(\theta,\varphi)$ becomes:

$$7) \quad f(\theta,\varphi) = -\frac{1}{4\pi}\int_{S'} \varphi(s') \frac{\partial}{\partial n'} e^{-ikr'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')]} ds'$$

Here r', θ', φ' are spherical coordinates on the surface S'. One may interpret $f(\theta,\varphi)$ by saying that $d\Omega \left| f(\theta,\varphi) \right|^2$ is the power scattered into an element of solid angle $d\Omega$ around the direction θ, φ. The backward direction is given by $\theta = \pi$, for which $f(\pi, \varphi)$ is independent of φ and equal to:

$$f(\pi) = -\frac{1}{4\pi}\int_{S'} \varphi(s') \frac{\partial}{\partial n'} e^{ikx'} \quad ds' .$$

This expression may be shown to be equal within a constant to $I(\varphi)$, so the back-scattered amplitude may be calculated directly by considering $I(\varphi)$ stationary. This may be done by standard methods of the calculus of variations.

IV. Asymptotic Developments of Steady State Electromagnetic Fields.

1. During this past year new investigations were started on the subject of asymptotic expansions of steady state electromagnetic fields. These expansions express the steady state field in the form of a series, not necessarily convergent but yielding the field fairly closely if even only two or three terms of the expansion are used, provided however that the basic variable in the expansion be large. The basic variable in the expansions sought in this research is the reciprocal of the wavelength, so that for small wavelengths the first few terms of the expansion might give a close approximation to the steady state field.

Asymptotic expansions are not new to electromagnetics but their application is thus far not extensive. It is also not a new idea to use geometric optics solutions as approximations to steady state electromagnetic fields for small wave lengths. What is new in this investigation is the development of a comprehensive theory which will furnish asymptotic expansions in which the geometric optics solutions appear as the first term of the expansion. This theory would then show the validity of the geometric optics solution in any particular application, and by the use of more terms of the series afford a better approximation to the true field.

The following result is basic in this investigation. Consider a source of electromagnetic waves which has any distribution in space but which at time $t = 0$ suddenly rises from 0 amplitude to amplitude 1. It is not supposed that space is homogeneous. The field satisfying Maxwell's equation and arising from this source is called the pulse solution of Maxwell's equations or the pulse field. If one now fixes on a point in space there will be discontinuities in this pulse field at this point at particular times t_α. These are the instants at which either the directly

transmitted wave from the source or waves reflected from obstacles in space pass this point. The discontinuities arise, of course, from the fact that at any time t_α a new wave passes this point whose amplitude is zero at time less than t_α but some non-zero value at times greater than t_α. There will also be discontinuities in the derivatives of the pulse solution as well as in the pulse solution itself. The basic result is that the first term in the asymptotic expansion of any steady state electromagnetic field arising from a harmonic source having the same space distribution as the pulse is given by the discontinuities in the pulse solution itself. Moreover, the successive terms in the asymptotic expansion are given by the discontinuities of the successive derivatives of the pulse solution. Explicitly these formulas are as follows:

$$u = -\frac{4\pi}{\varepsilon}\vec{g} + \sum [E_o]_\alpha e^{ik\psi_\alpha} - \frac{1}{i\omega}\sum \left[\frac{\partial E_o}{\partial t}\right]_\alpha e^{ik\psi_\alpha} - \frac{1}{i\omega}\int_0^\infty \frac{\partial^2 E_o}{\partial \tau^2} e^{i\omega\tau} d\tau$$

$$v = + \sum [H_o]_\alpha e^{ik\psi_\alpha} - \frac{1}{i\omega}\sum \left[\frac{\partial H_1}{\partial t}\right]_\alpha e^{ik\psi_\alpha} - \frac{1}{i\omega}\int_0^\infty \frac{\partial^2 H_o}{\partial \tau^2} e^{i\omega\tau} d\tau$$

In these formulas, u and v are the vector electric and magnetic field respectively, $\vec{g}$ is the space distribution of an impressed electric field of the form $\vec{g}(x,y,z)f(t)$, and E_o and H_o are the electric, respectively magnetic fields of the pulse solution. The square brackets with a subscript α denote the jump in the enclosed quantity at time t_α. The functions ψ_α give the wave fronts at each point of space at successive times t_α through the condition that the equations $ct = \psi_\alpha(x,y,z)$ are the hypersurfaces in space-time corresponding to wave fronts.

The connection between the discontinuities in the pulse solution and geometrical optics is essentially that the discontinuities are those furnished by wave fronts and the behavior of the wave front of any signal is the behavior of the electromagnetic field as furnished by geometrical optics. This connection which is here utilized is not new (see Luneburg, R.: Mathematical Theory of Optics, Brown University, 1944) . In the paper now being planned on the theory of asymptotic solutions some of this older material will be included for completeness sake and for ready reference.

2. The following material represents a substantial sketch of a major application that will be made of the theory just described; that is, it will be shown how the asymptotic field transmitted by a lens can be obtained. The word lens here can be interpreted in the usual optical sense. The theory, however, is broader in that it applies to any medium bounded by two parallel planes separating two semi-infinite homogeneous media.

Since the general theory relates the asymptotic behavior of a steady state electromagnetic field to the discontinuities of the pulse solution and the first term of the development is essentially the geometrical optics solution to the field, it is understandable that the theory finds ready application to problems intensively studied in the well developed field of geometrical optics, where this much of the solution to the steady state field is known. A brief sketch of the application now follows:

It is to be understood that the material in the sketch presented below, while fairly extensive in itself, still requires considerable development, and that a final paper on the subject may not be forthcoming in quite a few months.

2.1 Let us first consider the problem of transmission of light through optical instruments from the point of view of the electromagnetic theory of light. The x,y,z space is divided into two half spaces, called object and image space. These two halfspaces $z < z_0$ and $z > z_1$ are separated by the optical instrument, i.e. by a medium $z_0 < z < z_1$ in which the material constants $\mathcal{E}$, μ, σ vary with x,y,z. We assume that $\mathcal{E} = \mu = 1$, $\sigma = 0$ in object and image space.

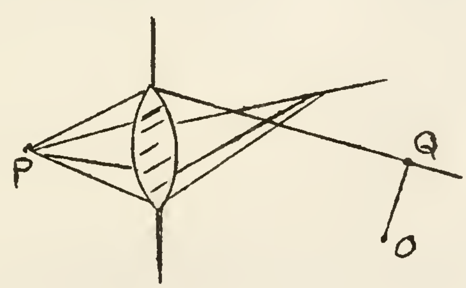

Fig. 1

A time periodic point source is located at a point P of the object space. The problem is to find the time periodic electromagnetic field

$$E = \vec{u}(x,y,z)\, e^{-i\omega t}$$
$$H = \vec{v}(x,y,z)\, e^{-i\omega t} \qquad (2.11)$$

which is established, in the image space, under the influence of this source. The vectors $\vec{u}$ and $\vec{v}$ then are solutions of the time free Maxwell's equations

$$\begin{aligned} \operatorname{curl} \vec{v} + ik\vec{u} &= 0 \\ \operatorname{curl} \vec{u} - ik\vec{v} &= 0 \end{aligned} \tag{2.12}$$

where $k = \dfrac{\omega}{c} = \dfrac{2\pi}{\lambda}$.

It is clear that the solution u,v of our problem must be somewhat related to the geometrical optics of our instrument, in particular to the system of light rays which originates at the point source P. This relation must be such that for small wave lengths, λ, i.e., for large values of $k = \frac{2\pi}{\lambda}$, the vectors u,v show an asymptotic behavior which expresses the geometric optical situation provided by the instrument. In other words we are concerned with a certain problem of correspondence: To find a solution u,v of (2.12) which, for $k \to \infty$, shows a prescribed asymptotical behavior.

2.2 We characterize the rays of the instrument by <u>Hamilton's mixed characteristic</u> $W(x_0, y_0, z_0; p, q)$. This is a function of the coordinates x_0, y_0, z_0 of the source and of the direction cosines p,q,r of the rays in the image space. It determines the optical length of the ray between the point P_0 and the base point Q of the perpendicular dropped from the origin O onto the ray p,q,r. We can derive all essential characteristics of the ray system with the aid of W. The ray of direction p, q, r, for example, is given by the equation

$$\begin{aligned} x - \frac{p}{r} z + W_p &= 0 \\ y - \frac{q}{r} z + W_q &= 0 \end{aligned} \tag{2.21}$$

If we calculate p and q from these equations as functions of x,y,z and introduce the result into the expression

$$\varphi = px + qy + rz + W \tag{2.22}$$

then a function $\varphi = \psi(x,y,z)$ is obtained which gives the optical length of a ray connecting the source P_o with the point x,y,z. The surfaces $\psi(x,y,z)$ = constant thus are the wave fronts of the system i.e., surfaces orthogonal to the rays.

There may, of course, exist several wave fronts through the same point x,y,z just as there may exist several rays which connect this point with the source P_o.

Let $d\omega$ be the surface element in which a narrow "tube" of rays intersects a wave front ψ = constant. We measure the expansion of the ray system by the expression

$$K = \frac{dp\,dq}{d\omega\sqrt{1 - p^2 - q^2}} \qquad (2.23)$$

which in our case of a homogeneous space is equal to the Gaussian curvature of the wave front.

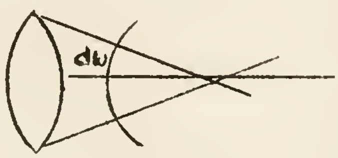

Fig. 2

In geometrical optics, light rays are considered as the carriers of light impulses. If the light distribution along a ray is observed then, according to geometrical optics, the polarisation of light is unchanged whereas intensity and energy flux vary proportional to the above quantity K. We may describe these observations mathematically by a vector field $\vec{U}(x,y,z)$; the direction of these vectors determines the polarisation of the light, the absolute value $|U|^2$ the energy and flux. The vectors $\vec{U}$ along a fixed ray p, q, r are normal to the ray and are related by an

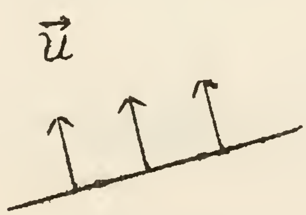

Fig. 3

equation

$$\vec{U} = \vec{A}(p,q)\sqrt{K} \qquad (2.24)$$

where the vector $\vec{A}(p,q)$ is constant along the ray.

2.3 The actual solution of the above transmission problem requires the integration of Maxwell's equations

$$\begin{aligned} \operatorname{curl} \vec{v} + ik\varepsilon \vec{u} &= 0 \\ \operatorname{curl} \vec{u} - ik\mu \vec{v} &= 0 \end{aligned} \qquad (2.31)$$

where ε and μ are given functions of x, y, z. On account of the mathematical difficulties which oppose such an approach one prefers to establish solutions in the image space ($\varepsilon = \mu = 1$) by a hypothesis:

1) The vectors $\vec{u}$ and $\vec{v}$ are analytical solutions of (2.12) which are regular and uniformly bounded in the total x,y,z space.

2) The vectors $\dfrac{\vec{u}}{\sqrt{K}}\, e^{-ik\Psi}$ and $\dfrac{\vec{v}}{\sqrt{K}}\, e^{-ik\Psi}$ attain definite limit values $\vec{A}(p,q)$ and $\vec{B}(p,q)$ if infinity is approached along a ray p, q, r of the instrument. The vector $\vec{A}(p,q)$ is identical with the vector $\vec{A}$ which has been introduced in (2.24) in order to characterize the geometric optics of the instrument. In other words it is assumed that at infinity both the actual electromagnetic field and the geometric optical approximation have the same asymptotic behavior.

By application of Green's theorem it can be shown that the above hypothetical solution can be obtained by superposition of plane waves, namely by the integrals

$$\begin{aligned} \vec{u} &= \frac{1}{2\pi i}\iint_{\Gamma} \vec{\mathfrak{A}}(p,q)\, e^{ik(px + qy + rz + W(p,q)}\, \frac{dp\,dq}{\sqrt{1 - p^2 - q^2}} \\ \vec{v} &= \frac{1}{2\pi i}\iint_{\Gamma} \vec{\mathfrak{B}}(p,q)\, e^{ik(px + qy + rz + W(p,q)}\, \frac{dp\,dq}{\sqrt{1 - p^2 - q^2}} \end{aligned} \qquad (2.32)$$

the domain of integration, Γ, is determined by the aperture of the instrument. It is a domain of the p, q plane which lies inside the unit circle $p^2 + q^2 = 1$. The

integrals (2.32) are solutions of the equations (2.12) for any choice of vectors $\vec{A}$, $\vec{B}$ provided that the conditions

$$\begin{aligned} \vec{p} \times \vec{B} + \vec{A} &= 0 \\ \vec{p} \times \vec{A} - \vec{B} &= 0; \quad \vec{p} = (p,q,r) \end{aligned} \tag{2.33}$$

are satisfied.

2.4 The evaluation of the integrals (2.32) by known functions is possible only in a few simple cases. Thus we are led to the problem of determining the principal terms of these integrals by an asymptotic development with respect to a parameter proportional to the quantity $k = \frac{2\pi}{\lambda}$. The solution of this problem then should provide the answer to the question whether the integrals (2.32) solve the above formulated problem of correspondence to a prescribed situation of geometrical optics. The answer will be in the affirmative: The principal terms of the integrals (2.32) are given by the expression

$$\begin{aligned} u_o &= \mathcal{A} \frac{\sqrt{K}}{k} e^{ik\psi} \\ v_o &= \mathcal{B} \frac{\sqrt{K}}{k} e^{ik\psi} \end{aligned} \tag{2.41}$$

provided that the dimensionless quantity

$$\frac{\sqrt{K}}{k} \gg 1. \tag{2.42}$$

This result shows that an asymptotic development beginning with (2.41) must break down in the neighborhood of caustics or focal points where $K = \infty$.

2.5 We approach the problem of asymptotic development as follows: We introduce the vectors

$$\begin{aligned} \vec{P}(x,y,z,t) &= \frac{1}{2\pi} \iint\limits_{\psi < ct} \frac{\vec{\mathcal{A}}(p,q)}{\sqrt{1-p^2-q^2}} \, dpdq \\ \vec{Q}(x,y,z,t) &= \frac{1}{2\pi} \iint\limits_{\psi < ct} \frac{\vec{\mathcal{B}}(p,q)}{\sqrt{1-p^2-q^2}} \, dpdq \end{aligned} \tag{2.51}$$

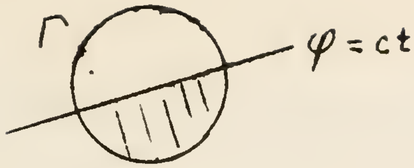

Fig. 4

The domain of integration is that part of the domain Γ where

$$\varphi = px + qy + rz + W < ct$$

We then may write the multiple integrals (2.32) in the form of simple integrals:

$$\vec{u} = -ic\int_0^\infty \vec{P}'(t)\, e^{i\omega t}\, dt$$
$$\vec{v} = -ic\int_0^\infty \vec{Q}'(t)\, e^{i\omega t}\, dt \qquad (2.52)$$

The vectors $\vec{P}'$ and $\vec{Q}'$ are, at any fixed point x,y,z, different from zero only in a finite interval $T_1 < t < T_2$, determined by the extreme values of the function $\frac{1}{c}\varphi$ in the domain Γ. In general $\vec{P}$ and $\vec{Q}$ will depend analytically on t; there are however certain points t_μ inside the above interval where $\vec{P}$ and $\vec{Q}$ or their derivatives are discontinuous. These points can be formed by studying the topology of the level lines φ = const. in Γ.

Consider for example a point x,y,z in the regular part of the field of rays (i.e., only one ray passes through x, y, z). In this case φ possesses exactly one stationary point, P_0, in the interior of Γ; The associated value, ct_0, is either a maximum or minimum of φ and equals the optical path $\psi(x,y,z)$ from the source to the point x,y,z. The level lines φ = constant thus surround the stationary point P_0 in the manner indicated in Fig 5. There exists two level lines

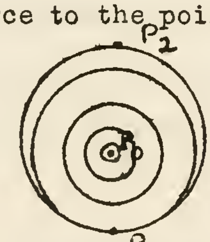

Fig. 5

$$\varphi = ct_1\ ,\quad \varphi = ct_2$$

(perhaps more) which are tangential to the boundary of Γ at points P_1 and P_2. We can find these level lines by determining the stationary points of the boundary values of the function φ. If A and B are analytic in Γ then $\vec{P}(t)$ and $\vec{Q}(t)$ will be analytic functions of t except at the points

$$t_0 = \frac{1}{c}\psi\ ;\ t_1 = \frac{1}{c}\psi_1\ ;\ \ t_2 = \frac{1}{c}\psi_2$$

given by the points P_o, P_1, P_2 of Fig. 5. The components of the vectors $\vec{P}$ and $\vec{Q}$ thus are functions of t of the type shown in Fig. 6.

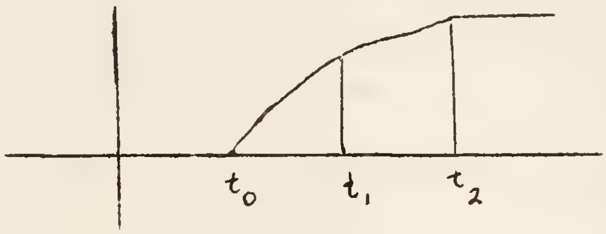

Fig. 6

If x,y,z lies outside the field of the rays then there is no interior stationary point of φ but again two or more stationary boundary points P_1 and P_2.

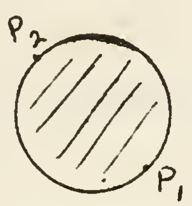

Fig. 7

Hence there are only two points

$$t_1 = \frac{1}{c}\,\psi_1\,(x,y,z); \quad t_2 = \frac{1}{c}\,\psi_2(x,y,z)$$

where $\vec{P}$ and $\vec{Q}$ are irregular.

If, finally, x,y,z lies in the irregular part of the ray system, i.e., if more than one ray passes through x,y,z, then there will be more than one stationary point in the interior and a corresponding number of stationary boundary points. In this case there exist a certain finite number of points $t_\mu = \frac{1}{c}\,\psi_\mu(x,y,z)$ where $\vec{P}$ and $\vec{Q}$ are non-analytic.

2.6 Let us now assume that $\vec{P}$ and $\vec{Q}$ are sectionally analytic functions of t, the different branches being separated by a discrete finite set of points

$$t_\mu = \frac{1}{c}\,\psi_\mu(x,y,z) \qquad ; \quad \mu = 1, \ldots\ldots N$$

We exclude these points by arbitrarily small intervals and transform the expressions (2.52) by repeated partial integration. We recognize that $\vec{u}$ and $\vec{v}$ admit an asymptotic development of the type

$$\vec{u} = \sum_{\mu=1}^{N} U_\mu(x,y,z)\; e^{ik\psi_\mu}$$

$$\vec{v} = \sum_{\mu=1}^{N} V_\mu(x,y,z)\; e^{ik\psi_\mu} \qquad (2.61)$$

where the functions $\vec{U}_\mu$ and $\vec{V}_\mu$ depend only on the values of $\vec{P}(t)$ and $\vec{Q}(t)$ in the immediate neighborhood of $t = t_\mu = \frac{1}{c}\,\psi_\mu(x,y,z)$.

The asymptotic series of U_μ and V_μ with regard to the parameter, k, depends on the type of singularity of the vectors $\vec{P}$ and $\vec{Q}$ at the points, t_μ ; It is possible to derive the following theorem:

Let us assume that $P'(t)$ in the neighborhood of $t = t_\mu$ allows a representation of the following form

$$P'(t) = (t_\mu - t)^\alpha \sum_0^\infty P_n (t_\mu - t)^n \ ; \ t < t_\mu$$
$$P'(t) = (t - t_\mu)^\beta \sum_0^\infty Q_n (t - t_\mu)^n \ ; \ t > t_\mu \qquad (2.62)$$

Then the function U_μ in (2.61) may be developed into an asymptotic series of the form

$$U_\mu = \frac{c}{\omega} \sum_0^\infty \left[\frac{Q_n \Gamma(n+1+\beta)}{(-i\omega)^{n+\beta}} - \frac{P_n \Gamma(n+1+\alpha)}{(i\omega)^{n+\alpha}} \right] \qquad (2.63)$$

2.7 We have reduced the problem of asymptotic evaluation of the diffraction integrals to the problem of finding the character of the irregularities of the vectors $\vec{P}$ and $\vec{Q}$ at certain singular points t_μ. These singular points are determined by the system of the level lines φ = constant in Γ, in particular by the stationary points of the interior or of the boundary values of the phase φ. We mention that an interior regular maximum or minimum point of φ introduces an ordinary discontinuity ($\alpha = \beta = 0$) of $\vec{P}'(t)$, a stationary boundary point, however, a square root discontinuity $\alpha = 1/2$ or $\beta = 1/2$. This shows that the asymptotic contribution associated with stationary boundary points (P_1 and P_2 in Fig. 5) are of higher order in $\frac{1}{k}$ than those from interior stationary points. One also finds that the first coefficient P_o or Q_o , in case $\alpha = \beta = 0$, has the value $\vec{A}\sqrt{K}$ so that the principal term in the development (2.61) has the form

$$u_o = \frac{\vec{\mathfrak{a}}}{k}\sqrt{K}\ e^{ik\psi}$$
$$v_o = \frac{\vec{\mathfrak{B}}}{k}\sqrt{K}\ e^{ik\psi} \qquad (2.71)$$

as was stated above in (2.41).

If the function ψ possesses more than one interior stationary point then a finite sum of terms (2.71) is obtained as principal asymptotic contribution. This is the case if more than one ray passes through the point x,y,z in question.

Now it can be shown that the vectors $\vec{P}$ and $\vec{Q}$ are indeed the pulse solutions of Maxwell's Equations for this particular problem. The general theory requires that one know the pulse solution in order to determine the asymptotic behavior of the steady state field and this pulse solution is known explicitly in the form of the vectors $\vec{P}$ and $\vec{Q}$ for the particular problem of transmission through a lens that has been presented in this sketch.

3. There are several other items of research planned under this general study of asymptotic behavior of steady state fields. It may be possible to show that the general theory described under the first item above, namely the relationship between pulse solutions and steady state fields can be approached through general integral relationships which will avoid the difficulties now associated with the intricate use of complex variable theory. A second possible application and one long envisioned is to the problem of anomalous propagation, where the calculation of the fields can be represented by integrals of the same form as those occurring in the theory described above.

V. Inversion Method and Perturbation Method

1. The problem of propagation in a spherically symmetric medium was considered in an early Report [2] for a very special variation of index of refraction, It was assumed that the atmosphere of the earth consisted in a layer of finite height in which the index of refraction varies inversely as the square of the distance from the center of the earth. It is assumed that above this layer the index is constant. The earth is supposed poorly conducting.

It was recognised that the choice made of refractive index in the atmosphere did not correspond to physical reality but it was hoped that a method similar to that in the paper might be applied to a sufficient variety of refractive indexes to yield useful conclusions for the problem of anomalous propagation. This hope was not borne out and, in spite of the mathematical interest associated with certain aspects of the problem approached in this way, the general program came to be abandoned. In an early Progress Report submitted to the Laboratories there is some discussion of the types of variation in refractive index which can be handled by the method of the paper, and some numerical calculations showing their inadequacy to the problem.

The Report [2] uses the classical transformation of space known as inversion, given by $\rho = a^2/r$, where r is the distance of a point from the origin, ρ the distance to the origin of the transformed point collinear with the origin and the first point. The constant a represents the radius of the inversion-sphere, in this case the radius of the earth. The wave equation satisfied by the Herzian potential function in the case of constant index of refraction is transformed, by inversion, to an equation corresponding to an index varying according to the inverse square law. The complete solution of the transformed equation may be derived from the known solutions to the original equation. There remains, however, the difficulty of finding solutions consistent with the proper boundary conditions. A solution to the problem is obtained in the form of a slowly convergent series.

2. The problem of propagation in non-homogeneous media was investigated, for a time, from the point of view of certain modifications of classical perturbation methods. This attempt is discussed in a preceding Quarterly Progress Report.

VI. Wiener-Hopf Methods in Propagation Problems.

In past years, considerable progress had been made by members of the Radiation Laboratory, located at M. I. T., in the application of integral equations of the Wiener-Hopf type to problems in wave-guide and in propagation theory. It has not always been easy to know what class of problems remained that might be exploited by this or related methods. In connection with a recent expansion in the program undertaken by the group, one of the active members of the former Radiation Laboratory was engaged as consultant, and this class of problems is now being investigated by members of the group. So far, the activity has been primarily that of study and group discussion in a seminar devoted to this topic.

VII. Bibliography

In the course of study and exchange of ideas by members of the group, and through contact with other groups engaged in related research, there began to be collected a substantial Bibliography on the general topics of the propagation and scattering of electromagnetic waves. This was subsequently augmented by a systematic search through periodicals and references appearing in recent years. It is submitted as a Report [9].

VIII. Education

Serious attention was given to one of the objectives of government subsidization of university research, namely, the development of the scientific potential of the country. Considerable training in the field of electromagnetic radiation was afforded a few younger, pre-doctorate mathematicians, and much interest in the application of mathematics to electromagnetism stimulated among the students and faculty at New York University.

Two offerings in the Graduate School also aided in stimulating interest in the applications of mathematics to electromagnetic theory. During the academic year 1946-47 a seminar in electromagnetic problems and during the year 1947-48 an advanced course in electromagnetic theory were offered. Both were well attended. While these courses were not subsidized by government funds the incentive to offer them and the interest in them on the part of some of the students was motivated by association with the work of this group.

IX. Administrative Report (for 1 July to 30 September)

1. Personnel

Dr. Nathan Marcuvitz has been appointed to this project as consultant. His services are on a part time basis, commencing 1 July, 1948. Dr. Marcuvitz is Assistant Professor at Brooklyn Polytechnic Institute, and has a background of active participation on the theoretical staff of the Radiation Laboratory.

Dr. Samuel Karp has been appointed to this project as senior scientist. His services are on a full time basis, commencing 1 August, 1948. Dr. Karp has had extensive research experience at Brown University in mathematical techniques which we are seeking to apply.

2. Correspondence

Security. On 7 July, 1948, we received memo 345-1000-2 regarding security questionnaires from the Air Materiel Command, which we acknowledged requesting a supply of forms. On 13 July Lt. Col. M.C. Edenfeld called our attention to paragraph 2 subparagraph (b) of the Annual Secrecy Agreement and on 11 August Col. Theodore M. Natt, of Watson Laboratories, confirmed our understanding that in the case of non-aliens security clearance is required only for participation in projects of "secret" or "top-secret" classification.

On 23 August we were advised by Col. Theodore M. Natt that clearance is requested for subcontractors, where required under article 20.

We replied to Mr. S.S. Rosenberg, of Watson Laboratories, 26 August, that Doris A. Troy, 45 Astor Place, New York, and also the Duplicating Department of New York University do mimeographing for us. Correspondence is in progress to ascertain whether this constitutes subcontracting, in the meaning of the article 20.

Property

On 1 September we wrote to Lt. Henry W. Liljedahl, referring to earlier communications, in connection with the transfer of property charged to Contract No. W28-099-ac-170, and received reply that this property has been transferred to this Contract, No. W28-099-ac-172.

Miscellaneous

On 5 July we informed Mr. N. C. Gerson that Dr. Wilhelm Magnus, whose services for this project we had earlier solicited, was being brought to this country in connection with certain Navy contracts.

On 20 July we were advised by the Air Materiel Command that the University of Michigan had requested two copies of Research Report No. 172-7, entitled "Reflection and Transmission of Electromagnetic Waves by a Spherical Shell", by Herbert B. Keller and Joseph B. Keller. Permission was also requested to index the title, author, and abstract in the Air Technical Index (A.T.I.) catalog cards. We forwarded a copy of the above report to the Air Materiel Command on 30 July and expressed our desire to have this document indexed.

We have received Progress Reports of the Computation Laboratory of the National Applied Mathematics Laboratories for the months of June, July and August. Reports are also being received from the Cruft Laboratory of Harvard University and the Research Laboratory of Electronics at M.I.T.

3. Conferences

Several members of the group attended the Second Annual Colloquium on Applied Mathematics, held in Cambridge at M.I.T., 29-31 July. The Colloquium this year was devoted entirely to electromagnetics. Dr. R. Luneberg, of our group, was one of the invited speakers, unfortunately not able to participate by reason of illness.

Dr. Paul M. Marcus of the Office of Naval Research, visited this group on 10 August. We discussed the general nature of our work. It is hoped that Dr. Marcus who will travel in England on behalf of the O.N.R. may be able to assist us in establishing an interchange of ideas with workers in England on projects similar to ours.

Dr. Marvin Chodorov of Stanford University visited us on 15 August. We discussed the relationship of our work to the work being done at Stanford University under the direction of Professor W. W. Hanson.

X. Research Reports Nos. 172-1 to 172-9 submitted under Contract No.W28-099-ac-172

1. Reflection of Electromagnetic Waves, by Joseph B. Keller.
2. An Application of Inversion to Wave Propagation, by Bernard Friedman.
3. The Propagation of Electromagnetic Plane Waves in Plane Parallel Layers, by R. K. Luneberg.
4. Propagation of Dipole Radiation Through Plane Parallel Layers, by Jerome Lurye.
5. Reflection and Transmission of Electromagnetic Waves by Thin Curved Shells, Joseph B. Keller.
6. Propagation of Electromagnetic Waves from an Arbitrary Source Through Inhomogeneous Stratified Atmospheres, by R. K. Luneberg.
7. Reflection and Transmission of Electromagnetic Waves by a Spherical Shell, by Herbert B. Keller and Joseph B. Keller.
8. Maxwell's Equations in Spherically Symmetric Media, by R. K. Luneberg.
9. Bibliography: Propagation and Scattering of Electromagnetic Waves, compiled by Thelma Braverman and Jerome Lurye.